Supplementary Information:
Enhanced colloidal transport in twisted magnetic patterns

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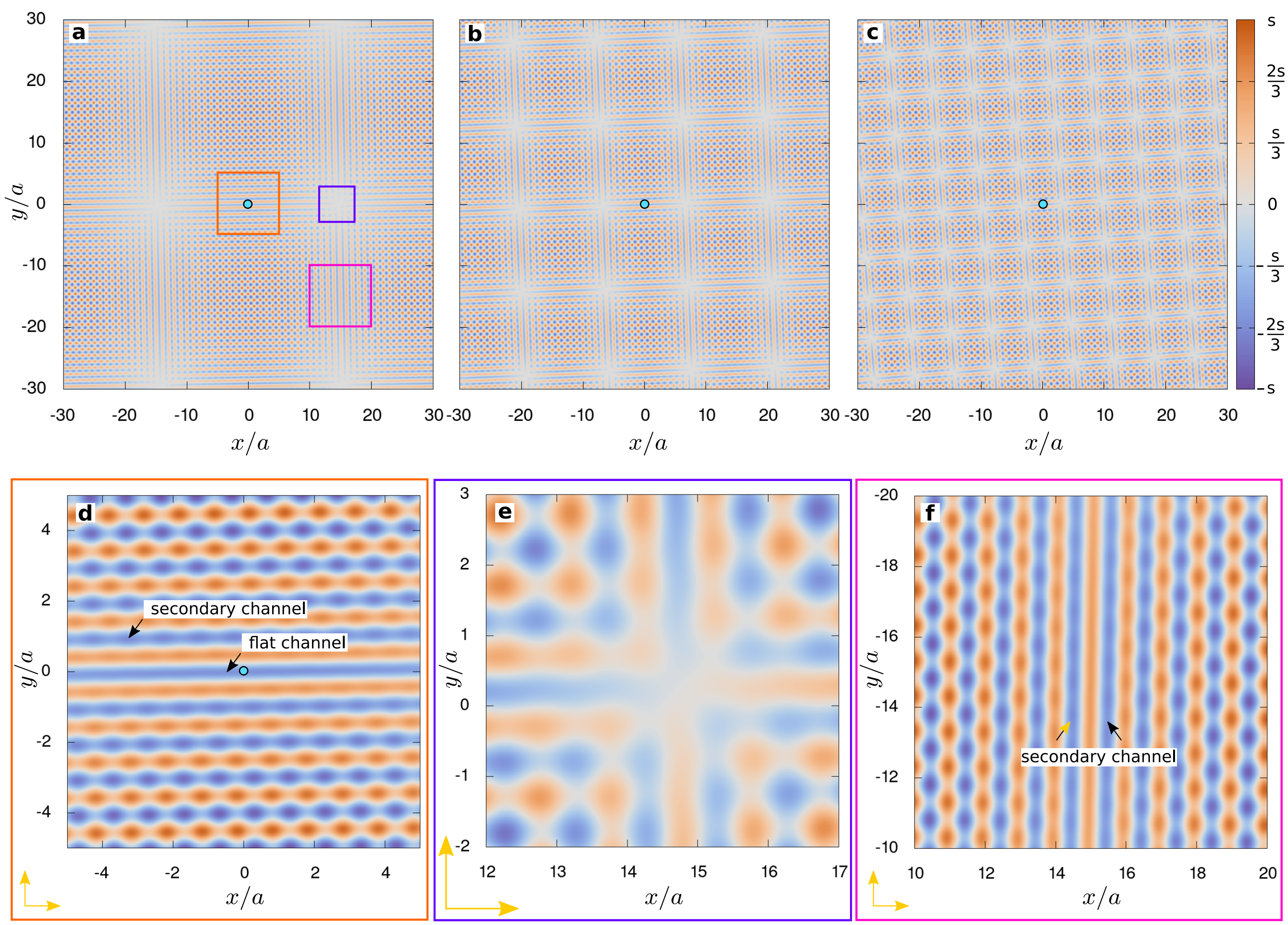
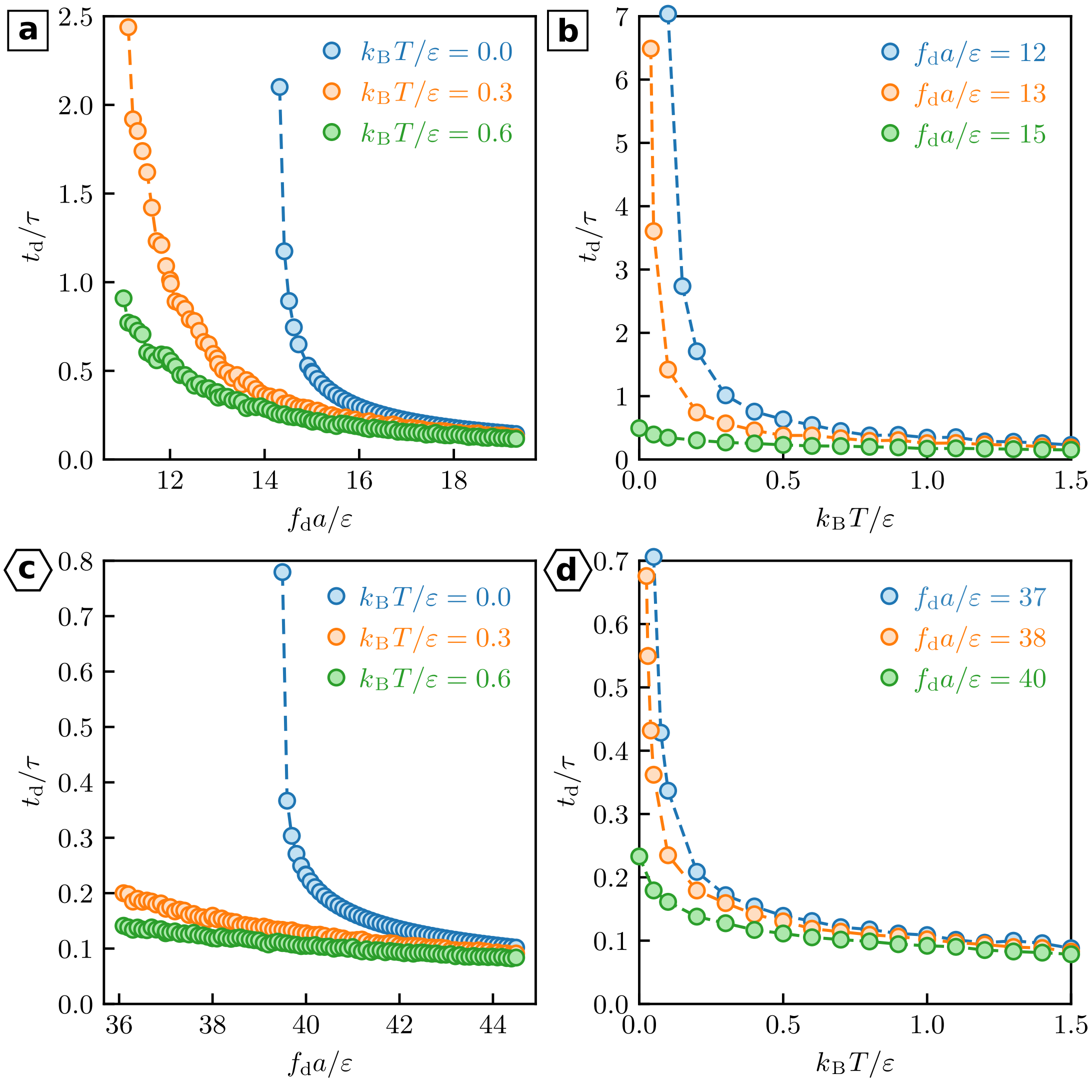
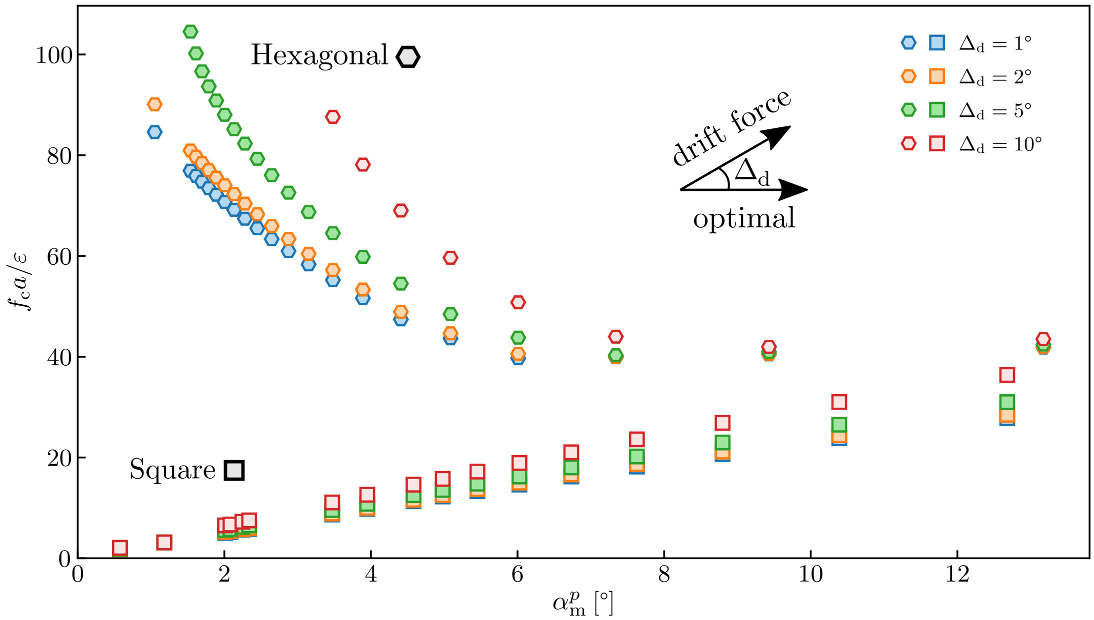


Supplementary Figure 1. **Twisted square patterns.** Magnetic potential, $V_{\rm mag}$, (color-coded) in twisted square patterns. One pattern is twisted around the axis normal to the patterns that passes through the origin (indicated by blue circles). The magic twist angle is $\alpha_{\rm m}^{\rm sq} \approx 2.01°$ in (a), $\alpha_{\rm m}^{\rm sq} \approx 4.24°$ in (b), and $\alpha_{\rm m}^{\rm sq} \approx 8.80°$ in (c). The untwisted pattern is shifted by $\mathbf{a}_1/2$. Panels (d), (e), and (f) are close views of selected regions for the magic angle $\alpha_{\rm m}^{\rm sq} \approx 2.01°$ (panel (a)), as indicated by the color boxes. The scale factor of the potential (see colorbar) is $s/\epsilon = 60$ in (a),(b), and (c), $s/\epsilon = 40$ in (d), $s/\epsilon = 15$ in (e), and $s/\epsilon = 40$ in (f). Panel (d) is a close view of a flat channel and the adjacent secondary channels (parallel to the flat channel and at a distance a of it), as indicated. Panel (e) is a close view of a corner that joins two flat channels. Panel (f) is a close view of an edge for which the potential is maximum. Two secondary channels, parallel to the edge and at a distance $a/2$ from it, are indicated. The lattice vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ of the untwisted pattern are parallel to the $x-$ and $y-$axes, respectively (see yellow arrows in panels (d),(e), and (f)).



Supplementary Figure 2. **Dwelling time at the corners.** Median time that a particle needs to cross the corner of a supercell t_d as a function of the magnitude of the drift force f_d at different temperatures, as indicated, in square (a) and in hexagonal (c) patterns twisted at a magic angle. Panels (b) and (d) show the crossing time t_d as a function of the temperature for different magnitudes of the drift force f_d, as indicated, in square and hexagonal twisted patterns, respectively. The magic angle is set to $\alpha_\mathrm{m}^\mathrm{sq} \approx 6.03^\circ$ and $\alpha_\mathrm{m}^\mathrm{hex} \approx 6.01^\circ$ in square and hexagonal patterns twisted at a magic angle, respectively. The crossing time for each individual particle is calculated by initializing the particle at position $\mathbf{x}_0^\mathrm{sq} \approx (3.9, 0.3)$ or $\mathbf{x}_0^\mathrm{hex} \approx (2.0, 0.2)$ for the square and hexagonal twisted patterns, respectively, and then measuring the time it takes the particle to cross a circle of radius $r^\mathrm{sq} = L/2$ or $r^\mathrm{hex} = L/3$. Here, $L = a/(2\sin(\alpha/2))$ and the circles are centered at the origin (axis of rotation). The initial positions are those of the the last local minima in the magnetic potential before the corners of the supercell. The circles of radii $r^\mathrm{p}, \mathrm{p} \in \{\mathrm{sq}, \mathrm{hex}\}$, cut through the corner of the supercells. When a particle has crossed the aforementioned circle it accelerates away from the corner of the supercell. To estimate t_d we take the median of 1000 individual crossing times. The strongest drift forces in panels (b) and (d), represented by green circles, are above the critical drift force at zero temperature and therefore the crossing time remains finite at $T = 0$. The other two values of the magnitude of the drift forces in (b) and (d), represented by blue and orange circles, are below the critical drift force at zero temperature and hence the curves diverge at zero temperature. Dashed lines are guides for the eye.



Supplementary Figure 3. **Effect of the direction of the drift force.** Magnitude of the critical force f_c required to transport particles at $T = 0$ as a function of the magic angle α_m^p in twisted square (square symbols) and hexagonal (hexagonal symbols) patterns for different directions of the drift force Δ_d, as indicated by the color of the symbols. The direction of the drift force is given by Δ_d which is the angle (in degrees) between the applied drift force and the optimal drift force. The optimal drift force points along the average direction between two consecutive flat channels.

SUPPLEMENTARY NOTE 1

We show here that for any non-magic angle the potential along the flat channels is at some point be unfavorable for the colloidal transport. To this end, we describe the local properties of patterns twisted at an arbitrary angle using the global properties of patterns twisted at angles for which the resulting potential is periodic.

Laboratory and local reference frames. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ designate the shift vectors of patterns 1 and 2, respectively, and let $\mathbf{f}$ denote the fixed point of the rotation. That is, the patterns are rotated relative to each other a total twist angle α about an axis normal to the patterns that passes through $\mathbf{f}$. In the main paper, we fix a laboratory reference frame in which $\mathbf{f} = \mathbf{0}$ (origin) and the shift vectors are $\mathbf{s}_1 = \mathbf{a}_1/2$ and $\mathbf{s}_2 = \mathbf{0}$. For our purpose here, we use also a local reference frame in which the fixed point is at the desired position $\mathbf{f} = \mathbf{r}$ and the shift vectors are hence position-dependent $\mathbf{s}_1(\mathbf{r})$, $\mathbf{s}_2(\mathbf{r})$. Shifting the patterns by the shift vectors in the local reference frame and twisting around the local fixed point generates the same potential as if we shift the patterns by the laboratory shift vectors and twist around the origin. Locations $\mathbf{r}$ and $\mathbf{r}'$ with similar shift vectors in the local reference frame, i.e at positions $\mathbf{r}$ and $\mathbf{r}'$, share a similar magnetic potential.

Periodic potentials and magic angles. The potential generated by two twisted patterns is periodic for any values of $\mathbf{f}, \mathbf{s}_1$, and $\mathbf{s}_2$ provided that the twist angle is [1, 2]

$$\alpha_{\mathrm{per}}^{(n)}(q/p) = 2\arctan\left(\frac{q/p\sin(\pi/n)}{1+q/p\cos(\pi/n)}\right), \tag{S1}$$

where p and q are co-prime integers, $n = 2$ for square patterns, and $n = 3$ for hexagonal patterns. Although the resulting potential is periodic for any pair of co-primes p and q, the magic angles are only those for which $q/p = 1/(nk+1)$ with $k \in \mathbb{N}$ (see Methods). Hence $\alpha_{\mathrm{m}}^{p} = \alpha_{\mathrm{per}}^{(n)}(1/(nk+1))$. For these values, the unit cell of the twisted patterns is the smallest one in terms of supercells: The unit cell of patterns twisted at magic angles is twice the size of the supercells in square patterns and of the same size in hexagonal patterns (see Fig. 2 of the main paper). For any other co-primes p and q in Eq. (S1) the potential is also periodic but the unit cell of the twisted patterns contains more (slightly different) supercells and, as we see below, the potential along the flat channels is at some point unfavorable for the transport.

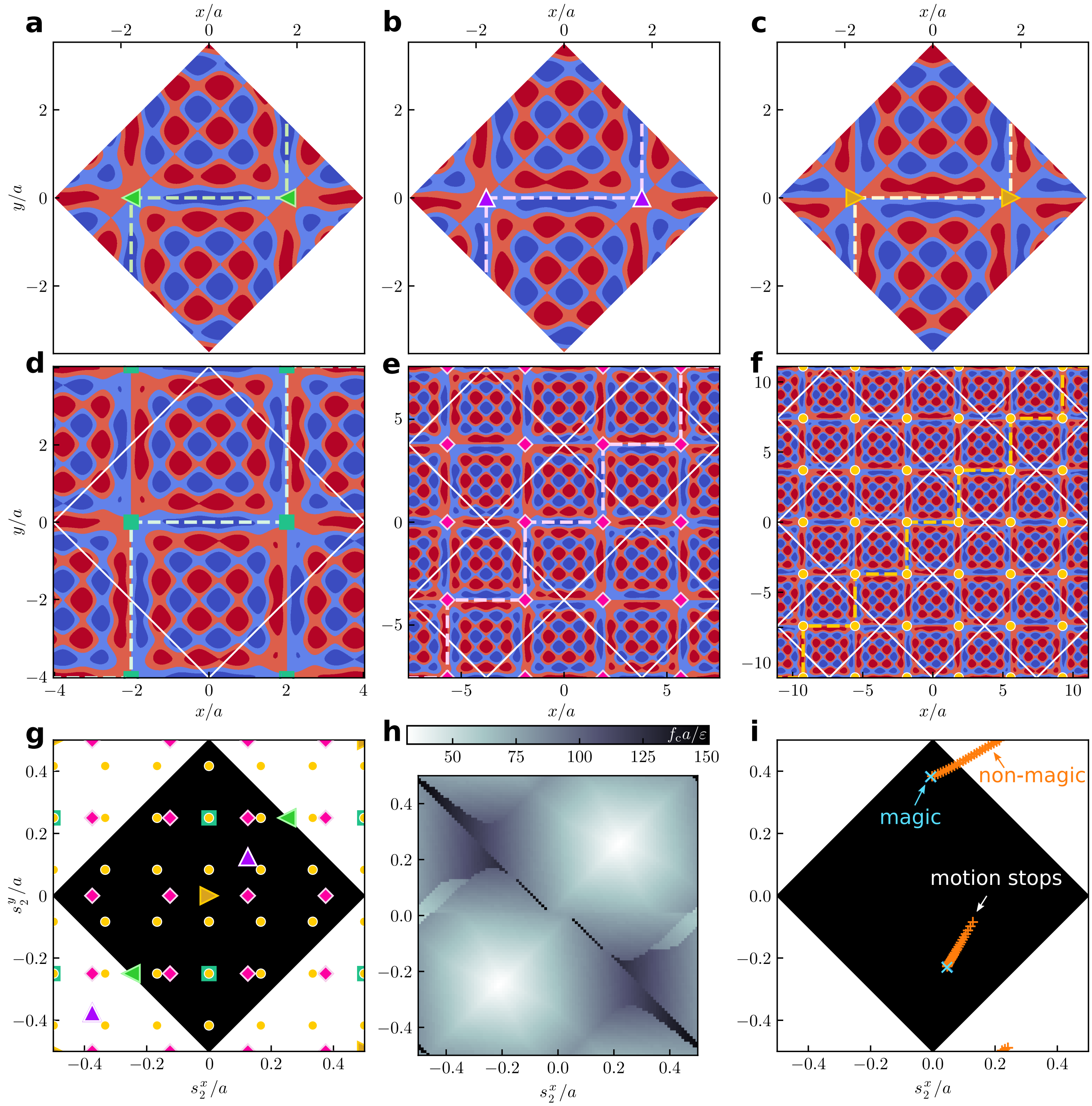
An arbitrary twist angle α can be approximated by an angle $\alpha_{\mathrm{per}}^{(n)}(q/p)$ using sufficiently large integers p and q. Hence, understanding transport in the periodic potentials given by twist angles $\alpha_{\mathrm{per}}^{(n)}(q/p)$ is enough to understand the transport for arbitrary twist angles.

Critical force at the corners of the supercells. In Supplementary Fig. 4 panels (a), (b), and (c) we show the magnetic potential in one unit cell of squared patterns twisted at magic angle $\alpha_{\mathrm{m}}^{\mathrm{sq}} = \alpha_{\mathrm{per}}^{(2)}(1/7) \approx 16.260°$ with fixed point at the origin ($\mathbf{f} = \mathbf{0}$) and for different shift vectors: $\mathbf{s}_1 = \mathbf{a}_1/2$ and $\mathbf{s}_2 = \mathbf{0}$ in panel (a), $\mathbf{s}_1 = 0.125(3\mathbf{a}_1 - \mathbf{a}_2)$ and $\mathbf{s}_2 = -0.125(\mathbf{a}_1 + \mathbf{a}_2)$ in panel (b), $\mathbf{s}_1 = 0.25(\mathbf{a}_1 - \mathbf{a}_2)$ and $\mathbf{s}_2 = -0.25(\mathbf{a}_1 + \mathbf{a}_2)$ in panel (c). We highlight with symbols (dashed lines) the position of the corners (edges) of the supercells.

Only one shift vector is enough to characterize specific locations of the pattern, such as the corners of the supercells, since there is a constraint that links both shift vectors at that specific location. For example, at the corners of the supercells $\mathbf{r} = \mathbf{r}_c$, the interference between both patterns is destructive and we get $\mathbf{s}_1(\mathbf{r}_c) - \mathbf{s}_2(\mathbf{r}_c) = \pm(\mathbf{a}_1 + \mathbf{a}_2)/n$ This equation for the position of the corners leads to a square (hexagonal) lattice that defines the supercells of the twisted pattern for any twist angle, independently of whether the angle is magic or not.

Knowing $\mathbf{s}_1(\mathbf{r}_c)$ in the local reference frame at the position of the corner fully determines $\mathbf{s}_2(\mathbf{r}_c)$ at the same position. Also, we can use the periodicity of the single patterns to fold the shift vectors into the unit cell of the single patterns. In panel (g) of Supplementary Figure 4 we indicate with the same symbols as in panels (a,b,c) the coordinates of the shift vector $\mathbf{s}_2(\mathbf{r}_c)$ at the corners of the supercells. Note that at magic angles the unit cell contains two supercells and hence only two sets of shift vectors at the corners exist [there are two triangles of each color in panel (g)]. In panel (h), we plot a color map of the critical force required to cross a supercell in the plane of the components of the shift vector $\mathbf{s}_2(\mathbf{r}_c)$ at the corners of the crossed supercell. In both panel (g) and panel (h) we have folded the shift vector $\mathbf{s}_2(\mathbf{r}_c)$ into the unit cell of a single pattern. The critical force has two global minima corresponding to the corners of patterns twisted at magic angles and shifted by half a unit vector [panel (a) and green triangles]. Again, these are the patterns discussed in the main text. For any other corner the critical force required to cross it increases.

Next, we consider patterns twisted at angles close to a magic angle and for which the potential is also periodic. We show in Supplementary Figure 4(d,e,f) the magnetic potential in one unit cell for three of such patterns. The twist



Supplementary Figure 4. **Transport at non-magic angles.** Magnetic potential in one unit cell of square patterns twisted at the magic angle $\alpha_{\mathrm{m}}^{\mathrm{sq}} = \alpha_{\mathrm{per}}^{(2)}(1/7)$ around $\mathbf{f} = \mathbf{0}$ and with shift vectors $\mathbf{s}_1 = \mathbf{a}_1/2$ and $\mathbf{s}_2 = \mathbf{0}$ in panel (a), $\mathbf{s}_1 = 0.125(3\mathbf{a}_1 - \mathbf{a}_2)$ and $\mathbf{s}_2 = -0.125(\mathbf{a}_1 + \mathbf{a}_2)$ in panel (b), $\mathbf{s}_1 = 0.25(\mathbf{a}_1 - \mathbf{a}_2)$ and $\mathbf{s}_2 = -0.25(\mathbf{a}_1 + \mathbf{a}_2)$ in panel (c). Panel (a) is the same type of twisted patterns as those discussed in the main text. The color map has been saturated to better visualize the regions of positive (red) and negative (blue) magnetic potential. The colored triangles indicate the position of the corners of the supercells. Panels (d), (e), and (f) show the magnetic potential in one unit cell of square patterns twisted around $\mathbf{f} = 0$ with shift vectors $\mathbf{s}_1 = \mathbf{a}_1/2$ and $\mathbf{s}_2 = \mathbf{0}$ [like in panel (a)]. The twist angles generate a non-magic but periodic potential: $\alpha_{\mathrm{per}}^{(n)}(q/p)$ with $q/p = 1/8$ in (d), $q/p = 2/15$ in (e), and $q/p = 3/22$ in (f). One unit cell contains 4 (d), 8 (e), and 32 (f) slightly different supercells (for visualization purposes, we draw with white lines pseudo unit cells, each containing 2 supercells). The symbols indicate the position of the corners of the supercells. The dashed lines illustrate the flat channel that would transport a particle located at the origin. (g) Components of the shift vector $\mathbf{s}_2$ at the corners of the supercells for the patterns shown in panels (a) to (f). Triangles correspond to patterns twisted at a magic angle (a,b,c) and other symbols represent patterns twisted at non-magic angles (d,e,f). The shift vector has been folded into the unit cell of a single square pattern. (h) Diagram of the critical force (color map) required to pass a supercell in the plane of the components of the shift vector $\mathbf{s}_2$. Data calculated with computer simulations in squared patterns twisted at $\alpha_{\mathrm{m}}^{\mathrm{sq}} = \alpha_{\mathrm{per}}^{(2)}(1/7)$. (i) Shift vectors (in the plane of the components of $\mathbf{s}_2$) of the trajectory points closest to the corners of supercells in square patterns twisted at magic angle $\alpha_{\mathrm{m}}^{\mathrm{sq}} = \alpha_{\mathrm{per}}^{(2)}(1/19) \approx 6.026°$ (blue crosses) and at non-magic angle $\alpha_{\mathrm{per}}^{(2)}(131457/2500000) \approx 6.020°$ (orange crosses). In the non-magic case the motion stops after 55 corners have been crossed. Data obtained for $T = 0$ and $f_{\mathrm{d}} a/\epsilon = 25$.

angles are given by Eq. (S1) with $q/p = 1/8 = 0.125$ in (d), $q/p = 2/15 \approx 0.133$ in (e), and $q/p = 3/22 \approx 0.136$ in (f). That is, in the three cases the angle is close to the magic twist angle $q/p = 1/7$ discussed above. The supercells are therefore of approximately the same size as in the magic case, e.g. panel (a), but since the angle is non-magic the unit cell contains several supercells (note the different scales used in the panels of Supplementary Figure 4). There are four (d), sixteen (e), and thirty two (f) supercells per each unit cell. The supercells slightly differ from each other, and therefore their corresponding shift vectors at the corners, shown in (h), are also different. Clearly, the shift vectors at the corners spread along the whole space instead of being concentrated around the region of small critical force (even though in the three cases $\mathbf{f} = \mathbf{0}$ and the shift vectors are $\mathbf{s}_1 = \mathbf{a}_1/2$ and $\mathbf{s}_2 = \mathbf{0}$, i.e. the shift vectors and the fixed point are like in the twisted patterns discussed in the main text). The higher the number of supercells per unit cell [i.e. the larger value of q in Eq. (S1)] the more spread the shift vectors are. This illustrates that transport in patterns twisted at magic angles is much more favorable than transport in periodic patterns twisted at angles that are non-magic.

Finally, to approximate a generic, non-periodic, twist angle [i.e. $\alpha \neq \alpha_{\rm per}^{(n)}(q/p)$] with arbitrary precision we need large values of p and q in Eq. (S1). However, increasing the value of q/p also increases the spread of the shift vectors at the corners of the supercells, which means that larger drift forces are required to pass all the supercells of a unit cell. To illustrate this phenomenon, we simulate the trajectories of particles moving in twisted patterns for which $q/p = 131457/2500000 \approx 0.05258$ and hence $\alpha_{\rm per}^{(2)}(131457/2500000) \approx 6.020°$. This value of q/p is very close to the magic case $q/p = 1/19 \approx 0.05263$, which we also simulate. For the magic case, $\alpha_{\rm m}^{\rm sq} = \alpha_{\rm per}^{(2)}(1/19) \approx 6.026°$. In the simulations, the particle trajectories never pass exactly through the corners of the supercells but close to them. For this reason, we plot in panel (i) of Supplementary Figure 4 the components of the shift vector $\mathbf{s}_2(\mathbf{r})$ corresponding to the trajectory points that are closest to the corners of the supercells for both the magic case (blue crosses) and the non-magic case (orange crosses). For the magic case we found only two points that correspond to the position of the minimal critical force (note again that the used shift vectors are not those at the corners but the closest ones along the trajectories). In contrast, in the non-magic case the shift vectors deviate more and more from the initial position until the particle gets stuck and the motion stops. Note that the motion stops when the shift vectors approach the diagonal along which the critical force to cross the supercells is large, see panel (h). If we want to sustain transport indefinitely, we would need to use a drift force that is able to cross the most unfavorable supercell. Similar phenomena (not shown) occurs for hexagonal twisted patterns.

SUPPLEMENTARY REFERENCES

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[1] S. Shallcross, S. Sharma, E. Kandelaki, and O. A. Pankratov, Electronic structure of turbostratic graphene, Phys. Rev. B **81**, 165105 (2010).
[2] T. Kariyado and A. Vishwanath, Flat band in twisted bilayer Bravais lattices, Phys. Rev. Res. **1**, 033076 (2019).