Blatt 3 - Hausaufgabe

Übung am 16. November 2018

Aufgabe 1: Nonconservative forces

a) D'Alembert's principle reads

$$\sum_i (\boldsymbol{F}_i - m_i \boldsymbol{a}_i) \cdot \delta \boldsymbol{r}_i = 0, \tag{1}$$

where $\boldsymbol{F}_i$ is the total applied force (excluding constraint forces) acting on the i-th particle, and $\delta \boldsymbol{r}_i$ is a virtual displacement of the i-th particle consistent with the constraints. Show that rewriting (1) in terms of the generalized coordinates $\boldsymbol{q}$, that is $\boldsymbol{r}_i = \boldsymbol{r}_i(\boldsymbol{q}, t)$, results in the following Lagrange equations (valid also for nonconservative forces)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial T}{\partial \boldsymbol{q}} = \boldsymbol{Q}, \tag{2}$$

where T is the total kinetic energy and Q_j is the generalized force associated to the generalized coordinate j, and it is given by

$$Q_j = \sum_i \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_j}. \tag{3}$$

Hint:

$$\frac{\partial \boldsymbol{r}_i}{\partial q_j} = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_j}, \qquad \frac{d}{dt}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial q_j}.$$

b) A single point mass m is attached to a rigid arm of negligible mass and length l. The mass is subject to a gravitational field and driven by a periodic non-conservative horizontal external force $\boldsymbol{f} = f_0 \cos(\omega t)\hat{\boldsymbol{x}}$. Find the equations of motion using both Eq. (3) and the Gibbs-Appell formulation.

Aufgabe 2: Nonholonomic constraints

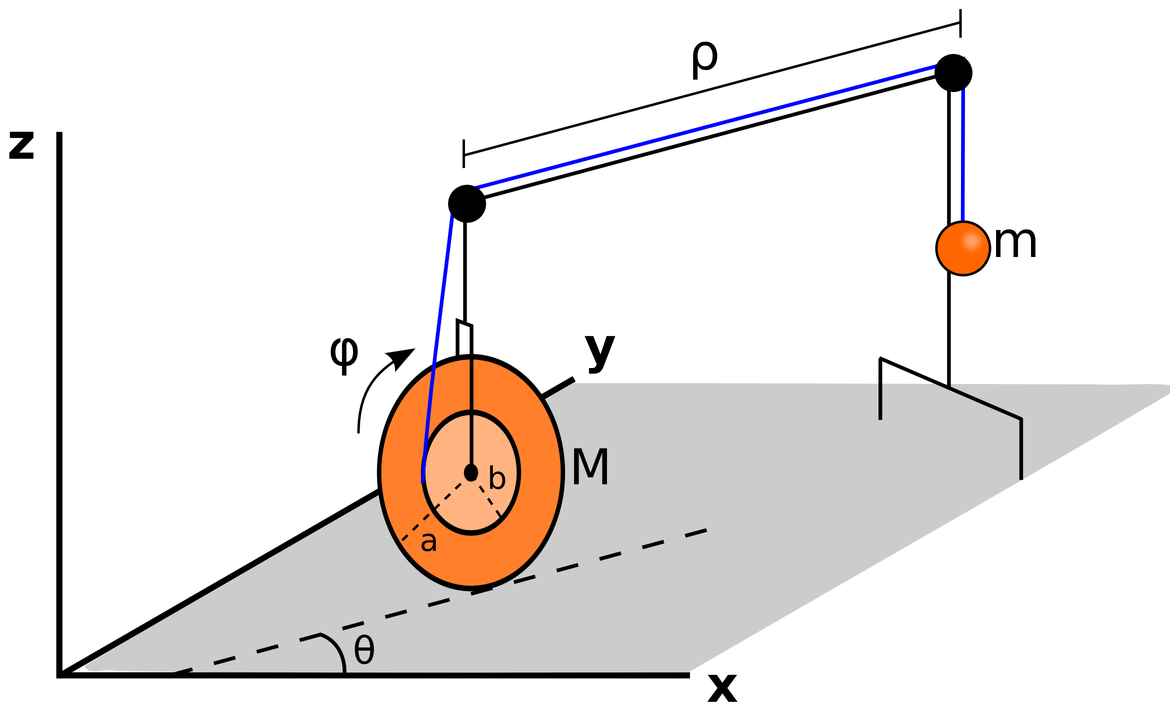
Consider the mechanical system depicted in the figure. A mass m is attached to the end of a massless string and restricted to move vertically. Using two massless pulleys separated by a distance ρ, the string is wound around a drum of radius b. The drum is attached to a wheel of radius a forming a rigid solid of total mass M. Due to the vertical descent of the mass, the wheel rolls without slipping on the horizontal xy plane. The moment of inertia of the system wheel-drum around the symmetry axis is I_1, and around an axis normal to the symmetry axis that passes through the center of the system is I_2. A massless frame with legs that slide without friction keeps the wheel vertical. An external force parallel to the direction tangent to the trajectory of the wheel, $\boldsymbol{f}_0 = (f_0 \cos\theta, f_0 \sin\theta)$, is applied to the mass m.

(a) Find the constraints of the motion.
(b) Show that for the limit $\rho \to 0$ the constraints can be written as

$$\begin{aligned} \dot{x}_m^2 + \dot{y}_m^2 &= \frac{a^2}{b^2}\dot{z}_m^2, \\ \dot{x}_m \sin\theta - \dot{y}_m \cos\theta &= 0. \end{aligned} \tag{4}$$

Here $\boldsymbol{r}_m(t)$ denotes the position of the mass.
(c) Use the Gibbs-Appell formulation to find the equations of motion if both $\rho \to 0$ and $I_1 \to 0$.
(d) Solve the equations of motion obtained in (c) for the component of the acceleration tangent to the trajectory of the wheel,

$$a_t = \frac{d}{dt}\sqrt{\dot{x}_m^2 + \dot{y}_m^2}. \tag{5}$$



ρ
z
m
φ
y
M
b
a
θ
x

Blatt 3 - Präsenzübung

Übung am 16. November 2018

Aufgabe 3: Nonholonomic constraints II

Find and sketch the trajectory of the center of mass of the disk in the Appell-Hamel problem (Blatt 3 - Aufgabe 2) in the limit $\rho \to 0$, $I_1 \to 0$ for the following cases:

(a)

$$\begin{aligned} x_m(t=0) = &= 0, \\ y_m(0) &= 0, \\ \dot{x}_m(0) &= 0, \\ \dot{y}_m(0) &= 0, \\ \theta(t=0) &= \theta_0, \\ \dot{\theta}(0) &= 0, \end{aligned}$$

(b)

$$\begin{aligned} x_m(t=0) &= 0, \\ y_m(0) &= 0, \\ \dot{x}_m(0) &= 0, \\ \dot{y}_m(0) &= 0, \\ \theta(t=0) &= 0, \\ \dot{\theta}(0) &= \omega, \end{aligned}$$